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EQUIVALENCE OF LCP AND PLS

by

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1. Introduction

It is our purpose here to show that two prototype models of complementary pivot and fixed point theory and the corresponding path following solution methods are conceptually equivalent. First, we have the linear complementarity problem

$$\text{(LCP)} \qquad Mz + Nw = q$$

$$z \geq 0 \qquad w \geq 0 \qquad z \cdot w = 0$$

where M and N are $n \times n$, q is $n \times 1$, and the variables z and w are $n \times 1$. Second, we have the piecewise linear system

$$\text{(PLS)} \qquad f(x) = y$$

where $f : R^n \to R^n$ is piecewise linear (PL). One attempts to solve LCP by following paths of solutions to the parameterized problem

$$\text{LCP}(\theta) \qquad Mz + Nw = Q(\theta)$$

$$z \geq 0 \qquad w \geq 0 \qquad z \cdot w = 0$$

where $Q : R_+ \to R^n$ is linear and $R_+ = [0, +\infty)$, this procedure is known as Lemke's algorithm [5].

One attempts to solve PLS by following paths of solutions to the parameterized problem

$$\text{PLS}(\theta) \qquad\qquad f(x) = Y(\theta)$$

where $Y : R_+ \to R^n$ is linear, see [4].

By converting an LCP to a PLS and vice-versa we shall demonstrate that paths following in one is conceptually equivalent to paths following in the other.

The conversion from LCP to PLS was introduced in [4] and is elementary. The conversion from PLS to LCP is the contribution of this paper and is comparatively complicated. It was shown in Lemke [6] that the primitive set schema of Scarf [10] could be posed as an LCP. Although we shall not pursue the matter here, the refining grid homotopy algorithms of [2, 3] and the restart algorithms of Merrill [8], assuming finite subdivisions, can also be shown equivalent to paths following in the LCP and PLS.

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2. Cells, Subdivisions, PL Maps

The remarks of this section are preliminary to the main development. By a cell we mean a closed polyhedral convex set σ, namely, a set of form $\sigma = \{x : Ax \leq a\}$. By an n-cell we mean a cell of dimension n. By a vertex, an edge, and a facet of a cell we mean faces of the cell of dimension 0, 1, and $n-1$, respectively. Two cells σ and τ are defined to be isomorphic if there is a linear map $h : \sigma \to \tau$ that is one-to-one and onto. In this case, in particular, $\dim \sigma = \dim \tau$ and τ is unbounded if and only if σ is. A cell is called pointed if it has a vertex.

A pointed cell σ can be expressed in the form

$$\{v\lambda + u\gamma : e\lambda = 1,\ \lambda \geq 0,\ \gamma \geq 0\}$$

where $v = (v_1, \ldots, v_\ell)$ in $R^{n\times\ell}$ is an ordering of the vertices v_i of σ, $\rho = \{\rho_1, \ldots, \rho_k\}$ is an ordering of the unbounded edges ρ_i of σ, $u = (u_1, \ldots, u_k)$ in $R^{n\times k}$ has columns u_j which are the (nonzero) directions (of recession) of the unbounded edges ρ_j of σ, $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in R^\ell$, $\gamma = (\gamma_1, \ldots, \lambda_k) \in R^k$, and $e = (1, \ldots, 1) \in R^\ell$. See Figure 1. For a given x in σ there may be many pairs $(\lambda, \gamma) \geq 0$ with $x = v\lambda + u\gamma$ and $e\lambda = 1$.

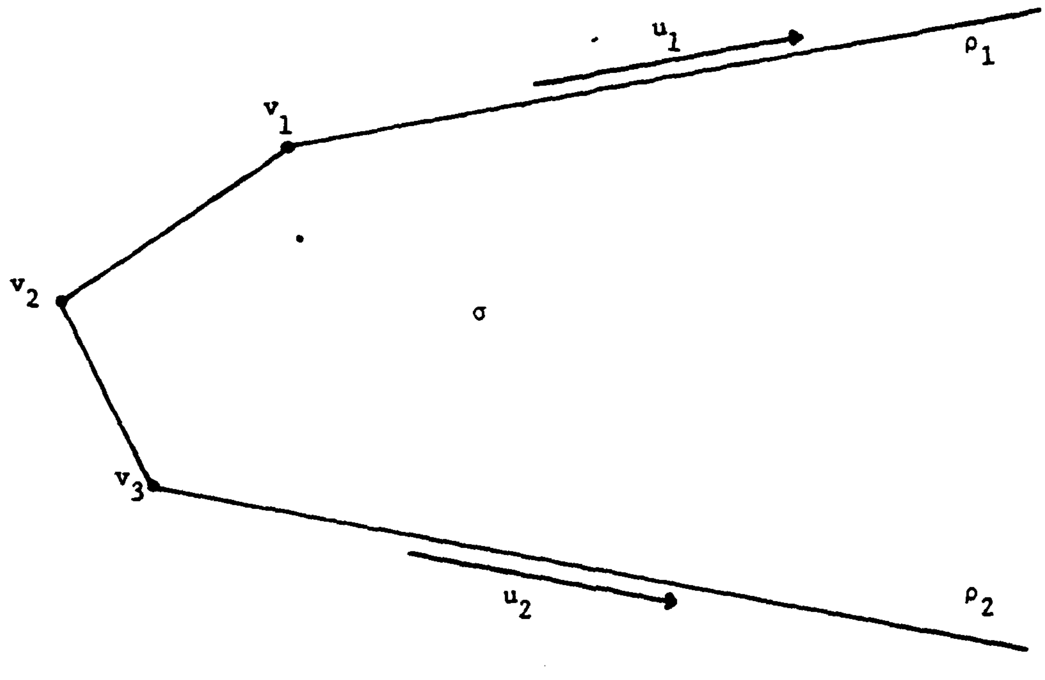

Figure 1

Let $f : \sigma \to R^n$ be linear and defined by $f(x) = Hx + h$. Then for any $x = v\lambda + u\gamma$ with $e\lambda = 1$, $\lambda \geq 0$, and $\gamma \geq 0$ we have $f(x) = V\lambda + U\gamma$ where $V = (f(v_1), \ldots, f(v_\ell))$ and $U = (Hu_1, \ldots, Hu_k)$.

By a subdivision $\mathcal{M}$ of a cell σ we mean a finite collection of cells τ contained in σ such that

1) the cells of $\mathcal{M}$ cover σ,

2) faces of cells of $\mathcal{M}$ are in $\mathcal{M}$

3) any two cells of $\mathcal{M}$ are disjoint or meet in a common face.

See Figure 2. The empty set is not regarded as a face of a cell and we do not include the empty set in a subdivision.

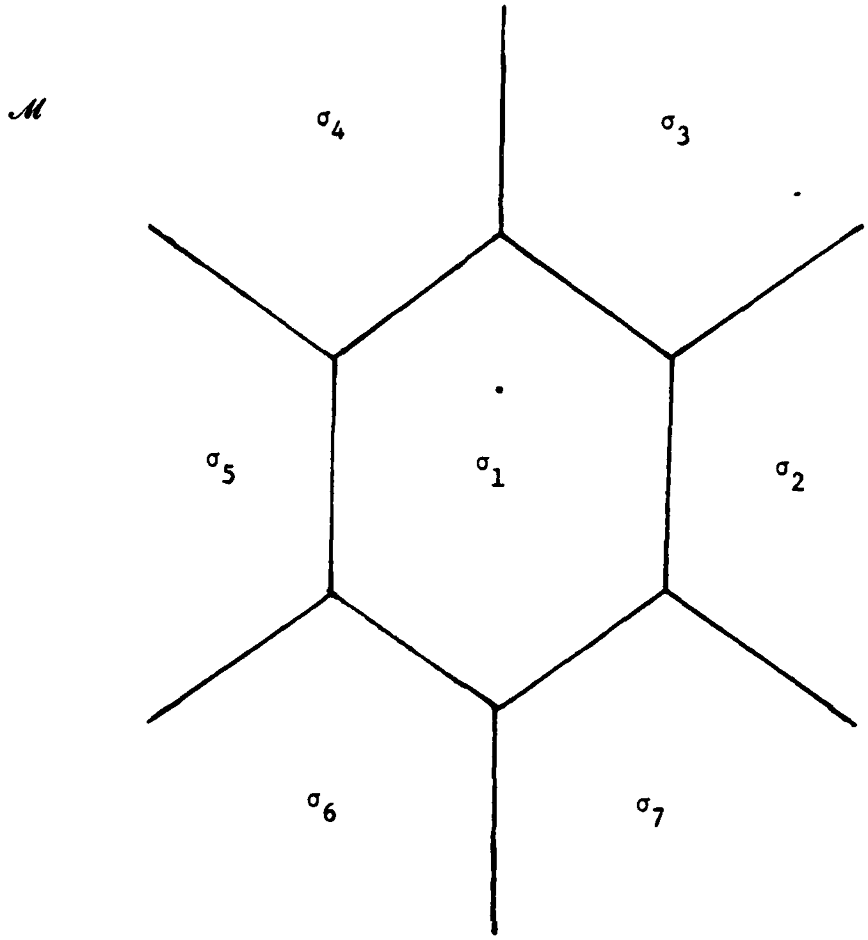

Figure 2

Given two subdivisions $\mathscr{M}$ and $\mathscr{N}$ of σ we say that $\mathscr{N}$ is a refinement of $\mathscr{M}$ if for each τ in $\mathscr{N}$ there is a ρ in $\mathscr{M}$ with $\tau \subseteq \rho$.

Let $\eta_1, \ldots, \eta_t$ be a finite collection of hyperplanes in R^n. For all $p = (p_1, \ldots, p_t)$ with p_i in $\{+, -\}$ define the cells

$$\tau_p = \bigcap_i \eta_i^{p_i}$$

where η_i^+ and η_i^- are the two closed halfspaces of R^n defined by the hyperplane η_i. Let $\mathcal{N}$ be the finite collection of n-cells of form τ_p. By extending $\mathcal{N}$ to include the faces of cells we obtain a subdivision of R^n and we define a subdivision of this form to be one based on hyperplanes. By adjoining more hyperplanes to the list we obtain a subdivision which refines the former, and in particular, by adjoining the hyperplanes $\{x : x_i = 0\}$ we obtain a refinement in which each cell is pointed.

2.1 Lemma: Let $\mathcal{M}$ be a subdivision of a cell. If one cell of $\mathcal{M}$ is pointed, then all cells of $\mathcal{M}$ are pointed. ☒

2.2 Lemma: Let $\mathcal{M}$ be a subidvision of R^n, then there is a subdivision $\mathcal{N}$ of R^n based on hyperplanes which is a refinement of $\mathcal{M}$.

Proof: Let γ_i be any facet of a cell σ_i of $\mathcal{M}$ and let η_i be the unique hyperplane containing γ_i. The subdivision $\mathcal{N}$ based on the η_i is a refinement of $\mathcal{M}$. ☒

Let σ be a cell and $f : \sigma \to R^n$ be a function. If for some subdivisions $\mathcal{N}$ of σ, f is affine on each cell of $\mathcal{N}$, then we say that f is PL, or more precisely, that it is PL with respect to $\mathcal{N}$. Clearly, if f is PL with respect to $\mathcal{N}$, and $\mathcal{M}$ is a refinement of $\mathcal{N}$, then f is PL with respect to $\mathcal{M}$.

By a vertex, unbounded edge, or direction of an unbounded edge of a subdivision $\mathcal{N}$ we mean such with respect to any cell of the subdivision. Let us assume that each cell of $\mathcal{N}$ is pointed. Let $v = (v_1, \ldots, v_\ell)$ in $R^{n\times\ell}$ and $\rho = (\rho_1, \ldots, \rho_k)$ be an ordering of all vertices and unbounded edges of $\mathcal{N}$. Let $u = (u_1, \ldots, u_k)$ in $R^{n\times k}$ where u_j is the direction of ρ_j. Let (λ, γ) be an element of $R^{\ell+k}$ with $(\lambda, \gamma) \geq 0$ and $e\lambda = 1$. Let τ be a cell of $\mathcal{N}$; the pair (λ, γ) is defined to be τ-admissible if $\lambda_i > 0$ implies v_i is a vertex of τ, and $\gamma_j > 0$ implies that ρ_j is an unbounded edge of τ. In this case $x = v\lambda + u\gamma$ is in τ. The pair (λ, γ) is defined to be admissible if (λ, γ) is τ-admissible for some τ.

2.3 <u>Lemma</u>: For x in R^n, there is an admissible pair (λ, γ) with $x = v\lambda + u\gamma$. ⊠

By a path X in R^n we mean a PL map $X : R_+ \to R^n$. A path (Λ, Γ) in $R^{\ell+k}$ is defined to be admissible if $(\Lambda(\theta), \Gamma(\theta))$ is admissible for each θ in R_+.

2.4 <u>Lemma</u>: If X is a path in R^n, then there is an admissible path (Λ, Γ) in $R^{\ell+k}$ with $X = v\Lambda + u\Gamma$.

<u>Proof</u>: Let the 0-cells of the subdivision of R_+ be $0 = t_1, \ldots, t_h$. Let $s_1, \ldots, s_g$ be those $\theta \in R_+$ such that $X(\theta)$

enters a cell of $\mathcal{N}$, that is, $X(s_i) \in \tau$ but $X(t_s - \varepsilon) \notin \tau$ for all sufficiently small positive ε. Let $r_0 = 0 < r_1 < \cdots < r_m$ be an increasing sequence with $\{r_0, \ldots, r_m\} = \{t_1, \ldots, t_h\} \cup \{s_1, \ldots, s_g\}$ and let $r_{m+1} = r_m + 1$. See Figure 3. For each r_p $p = 0, \ldots, m + 1$ let τ_p be the smallest cell of $\mathcal{N}$ containing $X(r_p)$ and select (λ^p, γ^p) τ_p-admissible with $X(r_p) = v\lambda^p + u\gamma^p$. For $p = 0, \ldots, m - 1$ define $(\Lambda(\theta), \Gamma(\theta)) = (1 + p - \theta)(\lambda^p, \gamma^p) + (\theta - p)(\lambda^{p+1}, \gamma^{p+1})$ for $p \le \theta \le p + 1$. For $\theta \ge m$ let $(\Lambda(\theta), \Gamma(\theta)) = (1 + m - \theta)(\lambda^m, \gamma^m) + (\theta - m)(\lambda^{m+1}, \gamma^{m+1})$. (Λ, Γ) is an admissible path and $X = v\Lambda + u\Gamma$.$\boxtimes$

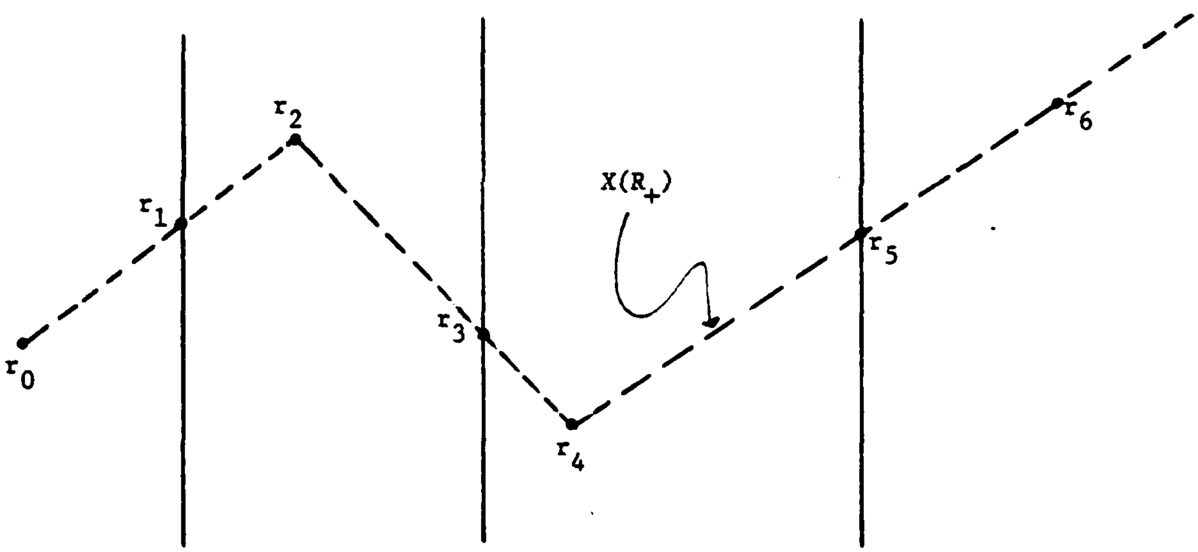

Figure $\underline{3}$

Let $f : R^n \to R^n$ be PL with respect to $\mathcal{N}$. Let $V = (f(v_1), \ldots, f(v_\ell))$ in $R^{n\times\ell}$ and $U = (h_1, \ldots, h_k)$ in $R^{n\times k}$ where if ρ_j lies in cell σ of $\mathcal{N}$ and $f(x) = Hx + h$ for x in σ, then $h_j = Hu_j$.

2.5 Lemma: If $x = v\lambda + u\gamma$ where (λ, γ) is admissible, then $f(x) = V\lambda + U\gamma$. ⊠

In Section 4 the theory is developed which enables us to state "admissibility" as a "complementary condition".

3. LCP to PLS

Since the transformation from an LCP to a PLS lends itself to a brief description we shall recount this conversion for completeness. In addition, this conversion gives the reader a paradigm for the PLS to LCP conversion which is considerably more involved.

Consider the LCP for $(z, w) \geq 0$ with $z \cdot w = 0$ let $x = z - w$. For x in R^n let $x^+ = (x_1^+, \ldots, x_n^+)$ and $x^- = (x_1^-, \ldots, x_n^-)$ where $x_i^+ = x_i$ if $x_i \geq 0$ and $x_i^- = -x_i$ if $x_i \leq 0$. Define $h : R^n \to R^n$ by

$$h(x) = Mx^+ + Nx^-$$

and it is clear that h is linear on each orthant of R^n and, hence, is PL. Note that this subdivision is based on the hyperplanes $\{x : x_j = 0\}$ and that each cell is pointed. Furthermore the x_j^+ and x_j^- are weights; that is, γ_j's, on directions, $\pm e_i$, of unbounded edges.

3.1 Theorem: If (z, w) solves LCP, then $x = z - w$ solves the PLS $h(x) = q$. If x solves the PLS $h(x) = q$, then (z, w) solves LCP where $z - w = x$ and $z \cdot w = 0$. ⊠

Recall that a path is a PL map on R_+. By the statement "(X, Θ) is a path solving $f(x) = Y(\theta)$", we mean that $f(X(t)) = Y(\Theta(t))$ for all t in R_+.

3.2 Theorem: If (Z, W, θ) is a path solving $LCP(\theta)$, then (X, θ) with $X = Z - W$ is a path solving the $PLS(\theta)$ $h(x) = Q(\theta)$. If (X, θ) is a path solving the $PLS(\theta)$ $h(x) = Q(\theta)$, then (Z, W, θ) is a path solving $LCP(\theta)$ where (Z, W) is defined by $X = Z - W$ and $Z \cdot W = 0$. ⊠

4. PL Convex Maps, Conjugates, and Duality

In this section we develop a duality relationship between the epigraphs of conjugate PL convex functions. This device is central to our conversion of PLS to LCP in the next section.

Let $g : R^n \to R^1$ be a convex function that is PL with respect to the subdivision $\mathcal{M}$ of R^n. Let G be the epigraph $\{(x, t) \in R^{n+1} : g(x) \leq t\}$ of g. Clearly G is an $(n+1)$-cell. Let $\mathcal{G}$ be the set of proper faces of G, that is, nonempty faces other than G itself. If the cells of $\mathcal{G}$ are projected to R^n then one obtains a subdivision $\mathcal{N}$ of R^n and $\mathcal{N}$ is a refinement of $\mathcal{M}$. Define $\psi : \mathcal{N} \to \mathcal{G}$ by $\psi(\tau) = \{(x, g(x)) : x \in \tau\}$. We say that ψ is inclusion preserving if $\sigma \subseteq \tau$ in $\mathcal{N}$ and $\psi(\sigma) \subseteq \psi(\tau)$ in $\mathcal{G}$ are equivalent.

4.1 Lemma: $\psi : \mathcal{N} \to \mathcal{G}$ is one-to-one, onto, inclusion preserving, and σ and $\psi(\sigma)$ are isomorphic for all σ in $\mathcal{N}$. $\boxtimes$

Let $g^* : R^n \to R^1 \cup \{+\infty\}$ be the conjugate, see Rockafellar [9] of g, namely,

$$g^*(y) = \sup(y \cdot x - g(x)) .$$

The effective domain $D = \{x : g^*(x) < +\infty\}$ is bounded and $g^* : D \to R^1$ is a convex PL function. Let G^* be the epigraph $\{(x, t) \in R^{n+1} : g^*(x) \leq t\}$ of g^* and let $\mathcal{G}^*$ be the set of proper faces of G^*.

We define two maps ψ_1 and ψ_2 that carry elements of $\mathscr{G}$ to elements of $\mathscr{G}^*$, see Figures 4 and 5. By $\partial g(x)$ we mean the set of subgradient of g at x, see [9].

Let τ be an element of $\mathscr{G}$. Note that if $(z, g(z))$ and $(w, g(w))$ are any two points in the relative interior of τ we have $\partial g(z) = \partial g(w)$. Further

$$\partial g(z) = \bigcap_{(x,g(x))\in\tau} \partial g(x) .$$

We define $\psi_1 : \mathscr{G} \to \mathscr{G}^*$ by

$$\Psi_1(\tau) = \{(y, g^*(y)) : y \in \partial g(z)\}$$

where $(z, g(z))$ is any point in the relative interior of τ. Recall $y \in \partial g(z)$, if and only if $z \in \partial g^*(y)$. Thus, we have

$$\psi_1(\tau) = \{(y, g^*(y)) : z \in \partial g^*(y)\}$$

or that $\psi_1(\tau)$ is the face of G^* corresponding to the supporting hyperplanes of g at z, therefore, ψ_1 is well-defined. We call ψ_1 inclusion reversing if $\sigma \subseteq \tau$ in $\mathscr{G}$ and $\Psi_1(\sigma) \supseteq \psi_1(\tau)$ in $\mathscr{G}^*$ are equivalent. Let $\mathscr{G}^*_b$ be the collection of bounded cells in $\mathscr{G}^*$. The next two theorems are closely related to the notion of dual cells.

4.2 <u>Theorem</u>: $\psi_1 : \mathscr{G} \to \mathscr{G}^*_b$ is one-to-one, onto, inclusion reversing, and $\dim \psi_1(\sigma) = n - \dim \sigma$ for all cells σ in $\mathscr{G}$.

<u>Proof</u>: That $\psi_1(\tau)$ is bounded follows from the boundedness of D. If τ_1 and τ_2 are distinct faces of G, then $\partial g(z_1) \neq \partial g(z_2)$ for $(z_i, g(z_i)) \in \text{ri } \tau_i$ and we have that $\psi_1(\tau_1) \neq \psi_1(\tau_2)$. If σ is a bounded proper face of G^*, then for some x we have $\sigma = \{(y, g^*(y)) : x \in \partial g^*(y)\} = \{(y, g^*(y)) : y \in \partial g(x)\} = \psi_1(\tau)$ where τ is the smallest face of $\mathscr{G}$ containing x. If $\tau_1 \supseteq \tau_2$ in $\mathscr{G}$, then $\partial g(z_1) \subseteq \partial g(z_2)$ where $(z_i, g(z_i)) \in \text{ri } \tau_i$ so $\psi_1(\tau_1) \subseteq \psi_2(\tau_2)$. Finally, $\dim \partial g(x) = n - \dim \tau$ for $(x, g(x))$ in the relative interior of τ. ⊠

Now let us assume, in addition, that the level sets $\{x : g(x) \leq t\}$ of g are bounded; then G^* is an $(n+1)$-cell. Let $\mathscr{G}_u$ and $\mathscr{G}^*_u$ be the set of unbounded cells of $\mathscr{G}$ and $\mathscr{G}^*$, respectively. Define $\psi_2 : \mathscr{G}_u \to \mathscr{G}^*_u$ by

$$\psi_2(\tau) = \psi_1(\tau) + (0, R_+)$$

ψ_2 carries unbounded cells of G to "vertical" cells of G^*.

4.3 <u>Theorem</u>: $\psi_2 : \mathscr{G}_u \to \mathscr{G}^*_u$ is one-to-one, onto, inclusion reversing, and $\dim \psi_2(\sigma) = n + 1 - \dim \sigma$ for all cells σ of $\mathscr{G}_u$. ⊠

4.4 <u>Lemma</u>: For τ in $\mathscr{G}_u$, $\psi_2(\tau) \supseteq \psi_1(\tau)$. For τ in $\mathscr{G}_u$ and σ in $\mathscr{G}$, $\psi_2(\tau) \supseteq \psi_1(\sigma)$ implies $\psi_1(\tau) \supseteq \psi_1(\sigma)$. For τ and σ in $\mathscr{G}_u$, $\psi_2(\tau) \supseteq \psi_1(\sigma)$ implies $\psi_2(\tau) \supseteq \psi_2(\sigma)$. ⊠

Let v_i $i = 1, \ldots, \ell$ be the vertices of $\mathscr{G}$ and ρ_j $j = 1, \ldots, k$ be the unbounded edges of $\mathscr{G}$. Let $\alpha \subseteq \{1, \ldots, \ell\}$ and $\beta \subseteq \{1, \ldots, k\}$.

4.5 Corollary: The facets of G^* are $\psi_1(v_1), \ldots, \psi_1(v_\ell)$, $\psi_2(\rho_1), \ldots, \psi_2(\rho_k)$. Furthermore, vertices v_i $i \in \alpha$ and edges ρ_j $j \in \beta$ lie in the same proper face τ of G if and only if all their corresponding facets $\psi_1(v_i)$ $i \in \alpha$ and $\psi_2(\rho_j)$ $j \in \beta$ meet in the face $\psi_1(\tau)$ of G^*.

Proof: $v_i \in \tau$ if and only if $\psi_1(v_i) \supseteq \psi_1(\tau)$. $\rho_i \subseteq \tau$ if and only if $\psi_2(\rho_i) \supseteq \psi_1(\tau)$. ⊠

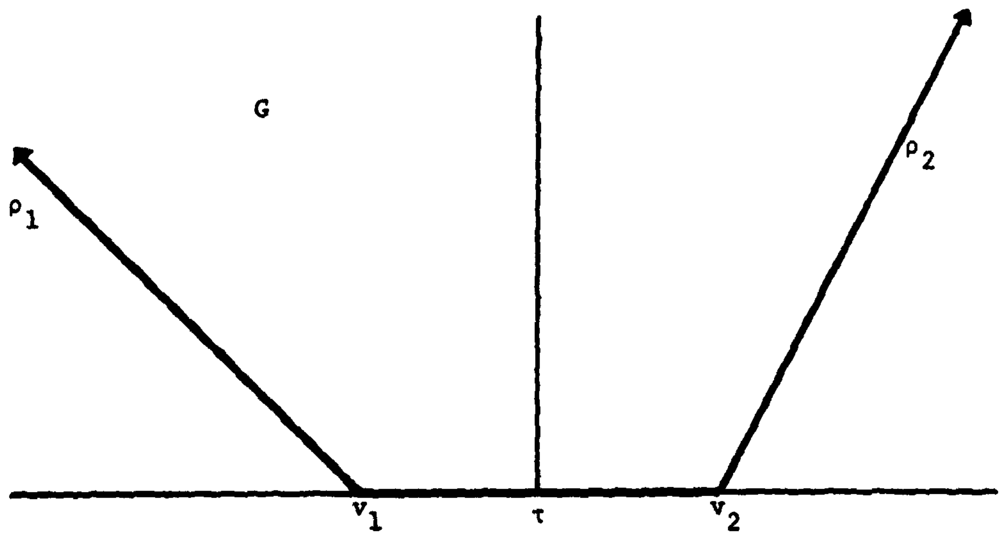

Figure 4a

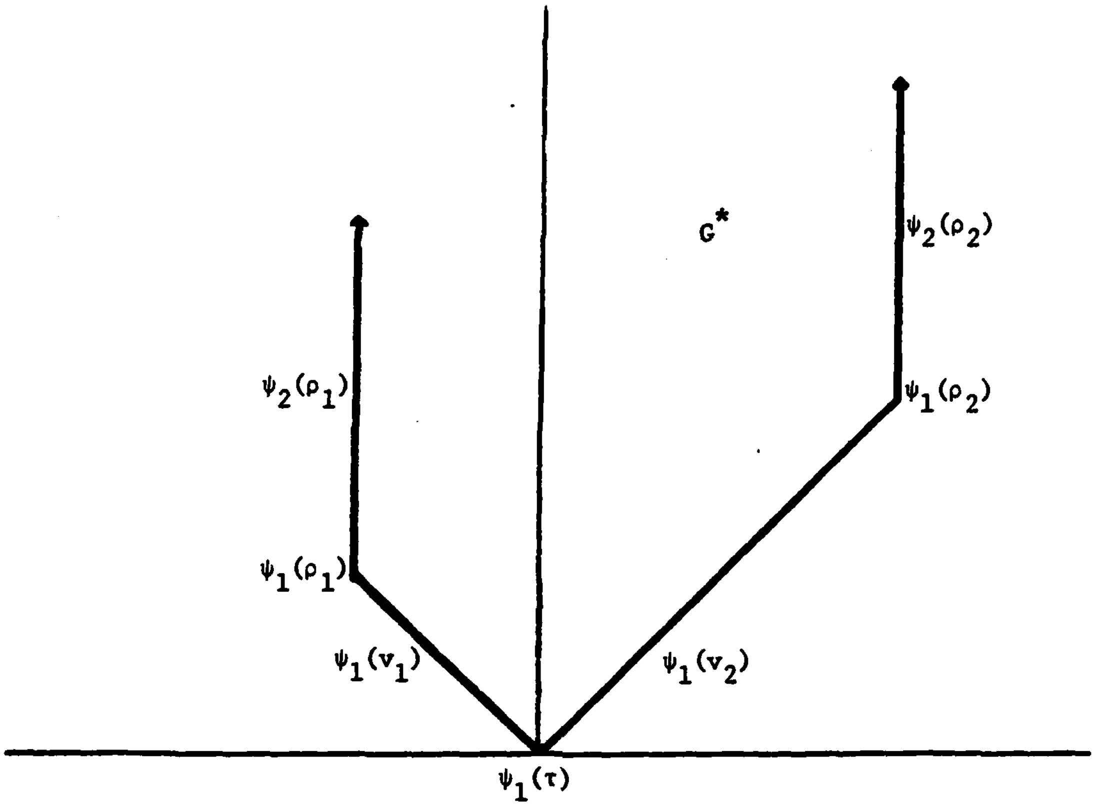

Figure 4b

$$
\begin{tikzcd}
\mathscr{G}_u \arrow[rr, leftrightarrow, "\psi_2"] \arrow[d, phantom, "\cap|"] & & \mathscr{G}_u^* \\
\mathscr{G} \arrow[rr, leftrightarrow, "\psi_1"] \arrow[d, leftrightarrow, "\psi"] & & \mathscr{G}_b^* \\
\mathscr{N} & &
\end{tikzcd}
$$

Figure 5

5. PLS to LCP

Consider the PLS. We can assume without loss of generality that f is PL with respect to a subdivision $\mathscr{N}$ that is based on the hyperplanes $\eta_1, \ldots, \eta_m$ and that each cell is pointed.

Let $v = (v_1, \ldots, v_\ell)$ in $R^{n\times\ell}$, $\rho = (\rho_1, \ldots, \rho_k)$, and $u = (u_1, \ldots, u_k)$ in $R^{n\times k}$ be an ordering of all vertices of $\mathscr{N}$, all unbounded edges of $\mathscr{N}$, and all directions u_j of the unbounded edges ρ_j. Let (λ, γ) be in $R^{\ell+k}$.

The crux of our conversion from PLS to LCP is to construct a cell E such that $(\lambda, \gamma) \geq 0$ with $e\lambda = 1$ is admissible if and only if $s \cdot (\lambda, \gamma) = 0$ for some s in E.

For $i = 1, \ldots, m$ define the "vee" maps $g_i : R^n \to R^1$ by

$$g_i(x) = |a_i x - b_i|$$

where $\eta_i = \{x : a_i x = b_i\}$. Then $g \equiv \sum_{i=1}^m g_i$ is a PL convex map. Let G be the epigraph of g and $\mathscr{G}$ be the collection of proper faces of G. As before $\psi : \mathscr{N} \to \mathscr{G}$ where $\psi(\tau) = \{(x, g(x)) : x \in \tau\}$ is one-to-one, onto, inclusion preserving, and σ and $\psi(\sigma)$ are isomorphic for all σ in $\mathscr{N}$. In particular, there is a one-to-one correspondence between the vertices and unbounded edges $(v, \rho) = (v_1, \ldots, v_\ell, \rho_1, \ldots, \rho_k)$ of $\mathscr{N}$ and those $(\psi(v), \psi(\rho)) = (\psi(v_1), \ldots, \psi(v_\ell), \psi(\rho_1), \ldots, \psi(\rho_k))$ of $\mathscr{G}$.

Since all cells of $\mathscr{N}$ are pointed the level sets of g are bounded and the epigraph G^* of g^*, the conjugate of g, is an (n+1)-cell. Define $\psi_1 : \mathcal{G} \to \mathcal{G}^*_b$ and $\psi_2 : \mathcal{G}_u \to \mathcal{G}^*_u$ as in Section 4. From corollary 4.5 we see that G^* has $\ell + k$ facets, one corresponding to each element of $(\psi(v), \psi(\rho)$, or equivalently, (v, ρ).

Thus, G^* can be written as the intersection of the $k + \ell$ halfspaces corresponding to its facets. In particular, we can express G^* as $\{\xi \in R^{n+1} : A\xi + Is = a,\ s \geq 0\}$ where the slacks $s = (s^1, s^2) = (s^1_1, \ldots, s^1_\ell, s^2_1, \ldots, s^2_k)$ of the facets are ordered to correspond to $(v, \rho) = (v_1, \ldots, v_\ell, \rho_1, \ldots, \rho_k)$.

Since G^* is pointed, A has rank $n + 1$. Thus, there are matrices (B, b) and (C, c) such that (ξ, s) solves $A\xi + Is = a$ with $s \geq 0$ if and only if (ξ, s) solves $\xi = Bs + b$, $Cs = c$, and $s \geq 0$. Furthermore C can be chosen to be $(k + \ell - n - 1) \times (k + \ell)$ and of rank $r + \ell - n - 1$. The set $E = \{s \geq 0 : Cs = c\}$ is the target of our effort as the next lemma shows.

We define the system * to be

$$* \begin{cases} Cs = c & e\lambda = 1 \\ (\lambda, \gamma, s) \geq 0 & (\lambda, \gamma) \cdot s = 0 \end{cases}$$

5.1 <u>Lemma</u>: (λ, γ) is admissible if and only if (λ, γ, s) solves * for some s. If (λ, γ) is σ-admissible and ξ is in $\psi_1\, \psi(\sigma)$ then $(\lambda, \gamma) \cdot s = 0$ where $s = a - A\xi$.

<u>Proof</u>: Suppose (λ, γ) is σ-admissible. Then the facets $\psi_1\psi(v_i)$ and $\psi_2\psi(\rho_j)$ corresponding to $\lambda_i > 0$ and $\gamma_j > 0$ meet

at the face $\psi_1\psi(\sigma)$. Select $\xi \in \psi_1\psi(\sigma)$ and let $s = a - A\xi$. Then $s_i^1 = 0$ for $\lambda_i > 0$ and $s_j^2 = 0$ for $\gamma_j = 0$, since ξ is in the facets $\psi_1\psi(v_i)$ and $\psi_2\psi(\rho_j)$ for $\lambda_i > 0$ and $\gamma_j > 0$. Suppose (λ, γ, s) solves $*$. Let $\xi = Bs + b$. Then the facets of G^* corresponding to $s_i = 0$ contain ξ and, hence, they contain the smallest face τ of G^* containing z. Thus, $\psi_1\psi(v_i) \supseteq \tau$ and $\psi_2\psi(\rho_j) \supseteq \tau$ for $\lambda_i > 0$ and $\gamma_j > 0$, respectively. Since $\lambda \neq 0$, τ is bounded. Therefore, $\psi_1\psi(\rho_j) \supseteq \tau$ according to lemma 4.4. Thus, v_i and ρ_j are in the cell $\psi^{-1}\psi_1^{-1}(\tau)$ for $\lambda_i > 0$ and $\gamma_j > 0$. Finally, $x = v\lambda + u\gamma$ lies in the cell $\psi^{-1}\psi_1^{-1}(\tau)$. $\boxtimes$

Let $V = (f(v_1), \ldots, f(v_\ell))$ and $U = (h_1, \ldots, h_k)$ where if ρ_i lies in cell σ of $\mathcal{N}$ and $f(x) = Hx + h$ for x in σ then $h_j = Hu_j$.

Now we define LCP_* to be the LCP

$$(LCP_*) \qquad M\binom{\lambda}{\gamma} + Ns = q$$
$$(\lambda, \gamma, s) \geq 0 \quad (\lambda, \gamma) \cdot s = 0$$

where $(M, N, q) =$

$$\left(\begin{array}{cc|c|c} 0 & 0 & C & c \\ V & U & 0 & y \\ e & 0 & 0 & 1 \end{array}\right)$$

LCP_* is a composite of $V\lambda + U\gamma = y$ and the system $*$.

5.2 <u>Theorem</u>: If (λ, γ, s) solves LCP_*, then $x = v\lambda + u\gamma$ solves PLS. If x solves PLS then * has a solution (λ, γ, s) with $x = v\lambda + u\gamma$ and any such (λ, γ, s) solves LCP_*.

<u>Proof</u>: Since (λ, γ) is admissible, $f(x) = V\lambda + U\gamma = y$ and x solves the PLS. If x solves the PLS, then x is in some cell τ of $\mathcal{N}$. So $x = v\lambda + u\gamma$ where (λ, γ) is τ-admissible and $f(x) = V\lambda + U\gamma = y$. Select ξ in the cell $\psi_1(\psi(\tau))$ and let $s = a - A\xi$. Then according to lemma <u>5.1</u> (λ, γ, s) solves *. $\boxtimes$

Let $LCP_*(\theta)$ be the system LCP_* with y replaced by $Y(\theta)$. In the theorem below $(\Lambda, \Gamma, S) : R_+ \to R^{\ell+k+(\ell+k)}$ and $\Omega : R_+ \to R_+$. Ω is defined to be monotone and onto if $s \leq t$ implies $\Omega(s) \leq \Omega(t)$ and $\Omega(R_+) = R_+$. Ω enters in the next theorem because the progress of (X, Θ) must be stopped periodically in order to update the complementary variable S corresponding to the subdivision. Heuristically, we alternate between a pivot corresponding to the function and a pivot corresponding to the subdivision (or in the language of Todd [11] between "primoid" and "duoid" exchanges). Recall, with a statement like "(X, Θ) solves $f(x) = Y(\theta)$" we mean that $f(X(t)) = Y(\Theta(t))$ for all t in R_+.

5.3 <u>Theorem</u>: If $(\Lambda, \Gamma, S, \Theta)$ is a path solving $LCP_*(\theta)$ then (X, Θ) solves $PLS(\theta)$ where $X = v\Lambda + u\Gamma$. If (X, Θ) is a path solving $PLS(\theta)$ then there is a path (Λ, Γ, S) and a monotone onto path Ω such that $(\Lambda, \Gamma, S, \Theta\Omega)$ solves $LCP_*(\theta)$ and $X\Omega = v\Lambda + u\Gamma$.

Proof: Since (Λ, Γ) is admissible $f(X) = V\Lambda + U\Gamma = Y\Theta$, see lemma 2.5. Now let (X,Θ) be a path solving PLS(θ). Let (Λ, Γ) be an admissible path with $v\Lambda + u\Gamma = X$, see lemma 2.4. Let $0 = \theta_0 < \theta_1 < \cdots < \theta_h < \theta_{h+1} = +\infty$ be a sequence in R_+ and $\sigma_0 \neq \sigma_1 \neq \cdots \neq \sigma_h$ a sequence in $\mathcal{N}$ such that $(\Lambda, \Gamma)(\theta)$ is σ_i-admissible for $\theta_i \leq \theta < \theta_i + 1$. Define $\Omega : R_+ \to R_+$ by

$$\Omega(\theta) = \begin{cases} (2i+1-\theta)\theta_i + (\theta - 2i)\theta_{i+1} & \\ \qquad \text{for } 2i \leq \theta \leq 2i+1 & i = 0, 1, 2, \ldots, h-1 \\ \theta_i \qquad \text{for } 2i+1 \leq \theta \leq 2i+2 & i = 0, 1, 2, \ldots, h-1 \\ \theta_h + \theta - 2h \quad \text{for } \theta \geq 2h & \end{cases}$$

See Figure 6. Let $S(2i) = a - Az$ for some z in $\psi_1\psi(\sigma_i)$ for $i = 0, 1, \ldots, h$. Let $S(\theta) = S(2i)$ for $2i \leq \theta \leq 2i + 1$ for $i = 0, 1, \ldots, h - 1$ and $\theta \geq 2h$. Thus, according to lemma 5.1 we have $(\Lambda, \Gamma)(\theta) \cdot S(2i) = 0$ for $\theta_i \leq \theta < \theta_{i+1}$ and $i = 0, \ldots, h_1$, or $(\Lambda, \Gamma)\Omega(\theta) \cdot S(\theta) = 0$ for $2i \leq \theta \leq 2i + 1$ for $i = 0, 1, \ldots, h - 1$ and $\theta \geq 2h$. Let $S(\theta) = (2i + 2 - \theta)S(2i + 1) + (\theta - 2i - 1)S(2i + 2)$ for $2i + 1 \leq \theta \leq 2i + 2$ for $i = 0, \ldots, h - 1$. Since the cells σ_i and σ_{i+1} meet in a common face τ, $\psi_1\psi\sigma_i$ and $\psi_1\psi\sigma_{i+1}$ are contained in $\psi_1\psi\tau$. Thus, $BS(\theta) + b$ is in $\psi_1\psi\tau$ for $2i + 1 \leq \theta \leq 2i + 2$. Since $(\Lambda, \Gamma)(\theta_{i+1})$ is τ-admissible we have $(\Lambda, \Gamma)(\theta_{i+1}) \cdot S(\theta) = 0$ for $2i + 1 \leq \theta \leq 2i + 2$ for $i = 0, \ldots, h - 1$. Or $(\Lambda, \Gamma)\Omega(\theta) \cdot S(\theta) = 0$ for

$2i + 1 \leq \theta \leq 2i + 2$ for $i = 0, \ldots, h - 1$. See Figure 7. So $(\Lambda, \Gamma)\Omega \cdot S = 0$. Clearly $v\Lambda\Omega + u\Gamma\Omega = X\Omega$ and $fX\Omega = V\Lambda\Omega + U\Gamma\Omega = Y\theta\Omega$. Therefore, $(\Lambda\Omega, \Gamma\Omega, S, \Theta\Omega)$ is the required path. ⊠

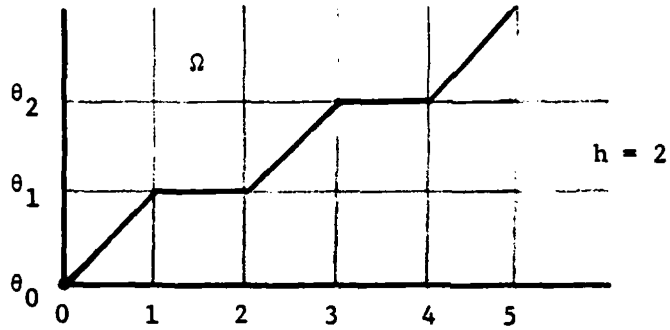

Figure 6

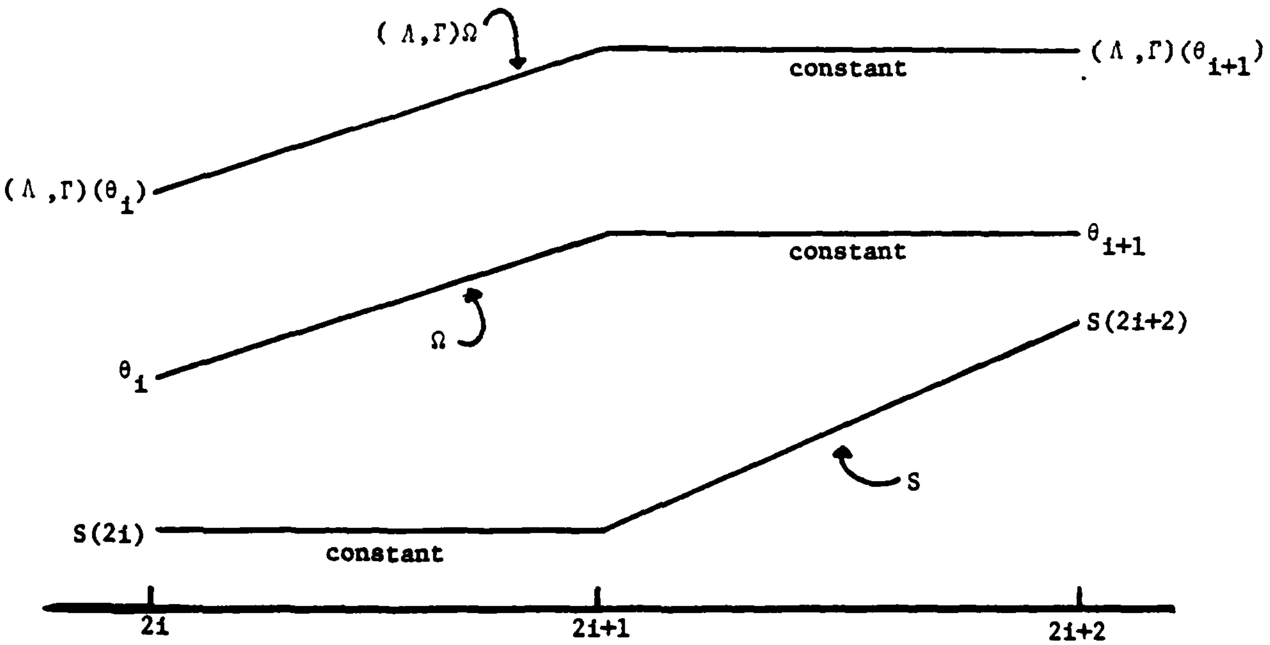

Figure 7

First, note that if we convert an LCP to a PLS, as in Section 3, and then convert the resulting PLS into an LCP, as in Section 5, we do not get back the original LCP. In particular, the dimension has expanded enormously. The conversion from PLS to LCP is, most likely, computationally useless, and consequently, perhaps, entirely useless. This conversion should, it seems, be regarded with certain suspicion; from a computational complexity perspective it is foolishness. Nevertheless, it is quite curious that, in fact, the PLS can be restated as an LCP in such a way that paths in PLS(Θ) transform to paths in LCP(Θ).

Second and continuing in the main theme, Aganagic [1] observed that RLCP can be stated on a convex PLS, namely, (CPLS) $h(z) = 0$ where $h : R^n \to R^n$ is convex and PL, and h is defined by

$$h_i(z) = \min (M_i z + q_i, z_i) \qquad \text{for} \qquad i = 1, \ldots, n$$

where M_i is the i^{th} row of M. But we have shown that PLS with f nonsingular somewhere can be restated as an RLCP. Thus, such a PLS can be transformed to a CPLS.

Third, our development was somewhat encumbered because the (λ, γ) coordinate system did not have unique representation. Is there some refinement of $\mathscr{N}$ that enjoys both unique representation as well as the existence of a set E?

Finally, if the function h in Section 3 is nonsingular on some orthant, then LCP can be transformed by simply rearranging the variables and multiplying by an inverse to obtain a regular LCP, RLCP, of the same size; that is, M and the identity I are $n \times n$.

$$\text{(RLCP)} \qquad Iw - Mz = q$$

$$z \geq 0 \quad w \geq 0 \quad z \cdot w = 0$$

In general if the map f of PLS is nonsingular on some n-cell σ where σ has no more than $n + 1$ vertices and unbounded edges, total, and hence, exactly $n + 1$, then LCP_* can also be so transformed to an RLCP. Thus, if f is nonsingular somewhere we can adjoin more hyperplanes so that f is nonsingular on an n-simplex of $\mathcal{N}$. Therefore, if f is nonsingular somewhere the PLS can be restated as an RLCP.

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The linear complementarity problem and the piecewise linear system are the two principal models in complementary pivot and fixed point theory. It is shown that the path following methods for solving each are conceptually equivalent.

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